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Existence of α-Open Loop Saddle Point in Fuzzy Games

Pramod Ola
Assistant Professor
Narayani Devi Girl's P.G. College, India

Abstract

In the present paper we analysis the solution of N-person zero sum continuous differential game in the fuzzy rough environment which uses the fuzzy rough sets to measure the dual and multiple uncertainties with high ambiguity and vagueness in continuous differential games. The combination of fuzzy and rough set in continuous differential games represents a new class defined as fuzzy rough continuous differential games. The necessary and sufficient conditions are produced and numerical example is presented to support the theoretical claims of N-person zero sum fuzzy rough continuous differential games.

Keywords: Fuzzy rough set, Fuzzy game, α-open loop saddle point, Continuous-time dynamic game.

Introduction

The traditional study of differential game models, developed under classical (crisp) mathematics, usually assumes that every player possesses complete and accurate information. Under this assumption, such games are analyzed as full-information systems. However, in many real-world applications, the information available to players is often imprecise or incomplete. These uncertainties are difficult to describe using purely deterministic or stochastic methods. To address this issue, such imprecision is represented as *fuzzy information*, and differential games incorporating this type of uncertainty are referred to as *fuzzy differential games*.

Differential game theory, initiated by Isaacs [9], has become a significant field of research due to its wide applications in economics, operations research, and management science. His foundational work focused mainly on two-player zero-sum dynamic game models. More recent studies have expanded on this framework. For example, Meghahed and Hegazy [11] examined a min–max two-player zero-sum continuous-time dynamic game with fuzzy control variables, while Hegazy [7] further explored solutions for such games when both controls and state trajectories are fuzzy.

Building on these developments, we extend the concept to multi-player zero-sum settings, in which one participant aims to maximize their performance index while all other players attempt to minimize theirs.
In addition, Pawlak's rough set theory [16] provides a mathematical tool to approximate subsets of a universe through lower and upper approximations. Zadeh's introduction of fuzzy set theory [20] further enriched the modelling of uncertainty. Several researchers have since combined fuzzy and rough set frameworks. Dubois and Prade [5] discussed various aspects of integrating these two theories, and Wygralak offered additional remarks on their interrelation.

Definitions and Preliminaries

Definition: A fuzzy set $\tilde{A}$ in R is called a fuzzy number if it satisfies the following properties:

Corresponding Author: Pramod Ola *pramodola1991@gmail.com*

[1] $\tilde{A}$ is normal. i.e. $\exists\ x_0 \in R$ s.t. $\mu_{\tilde{A}}(x_0) = 1$.
[2] For every α∈ (0,1] the α-cut of the fuzzy set $\tilde{A}$ is a closed interval.
[3] $The\ support\ set\ of\ \tilde{A}$ i.e. $S(\tilde{A})$ is bounded.$S(\tilde{A}) = \{x \in R\ ;\ \mu_{\tilde{A}}(x) > 0\}$.

Where $\mu_{\tilde{A}}$: The membership function takes values in the interval R [0,1] for a fuzzy number $\tilde{A}$."

Definition: "The α-level set or α-cut set of a fuzzy number$\tilde{A} \in F(R), 0 < \alpha \leq 1$, denoted by $(\tilde{A})_\alpha$ is closed and bounded interval $[A_\alpha^L, A_\alpha^U]$ where A_α^L and A_α^U denotes the left/right hand end point of $(\tilde{A})_\alpha$ respectively and defined as $(\tilde{A})_\alpha = \{x \in R\ ;\ \mu_{\tilde{A}}(x) \geq \alpha\}$, where $F(R)$ denote the set of all fuzzy numbers in R."

Definition: "The algebraic operations for two triangular fuzzy numbers $\tilde{A} = (a_1\ , a_2\ , a_3)$ and $\tilde{B} = (b_1\ , b_2\ , b_3)$ are defined as follows:

[1] $\tilde{A} + \tilde{B} = (a_1 + b_1,\ a_2 + b_2,\ a_3 + b_3)$
[2] $\tilde{A} - \tilde{B} = (a_1 - b_3,\ a_2 - b_2,\ a_3 - b_1)$
[3] $k\tilde{A} = (ka_1,\ ka_2,\ ka_3); k > 0$
[4] $-\tilde{A} = (-a_3, -a_2, -a_1)$."

Definition: "The α-cut for the triangular fuzzy number $\tilde{A} = (a_1,\ a_2,\ a_3)$ is defined as follows:
$(\tilde{A})_\alpha = [a_1 + (a_2 - a_1)\alpha,\ a_3 - (a_3 - a_2)\alpha]; \alpha \in (0,1]$."

Definition: "A fuzzy rough interval $\tilde{X}^R = [\tilde{X}^L : \tilde{X}^U]$ of a compact set *X* of real numbers satisfy following conditions,

[1] $\tilde{X}^R \geq \tilde{0}^R$ iff $\tilde{X}^L \geq \tilde{0}^L$ and $\tilde{X}^U \geq \tilde{0}^U$
[2] $\tilde{X}^R \leq \tilde{0}^R$ iff $\tilde{X}^L \leq \tilde{0}^L$ and $\tilde{X}^U \leq \tilde{0}^U$where $\tilde{X}^L$ is lower and $\tilde{X}^U$ is upper approximation fuzzy set."

Definition: "Let$\tilde{A}^R = [\tilde{A}^L : \tilde{A}^U]$, $\tilde{B}^R = [\tilde{B}^L : \tilde{B}^U]$s.t. $\tilde{A}^R \geq \tilde{0}^R$and $\tilde{B}^R \geq \tilde{0}^R$, then

[1] $\tilde{A}^R(+)\tilde{B}^R = [(\tilde{A}^L + \tilde{B}^L) : (\tilde{A}^U + \tilde{B}^U)]$
[2] $\tilde{A}^R(-)\tilde{B}^R = [(\tilde{A}^L - \tilde{B}^L) : (\tilde{A}^U - \tilde{B}^U)]$
[3] $\tilde{A}^R(\times)\tilde{B}^R = [(\tilde{A}^L \times \tilde{B}^L) : (\tilde{A}^U \times \tilde{B}^U)]$
[4] $\tilde{A}^R(/)\tilde{B}^R = [(\tilde{A}^L / \tilde{B}^L) : (\tilde{A}^U / \tilde{B}^U)]$."

Definition: "The point $(u_{1\alpha}^{*UU}(t), u_{2\alpha}^{*UU}(t), \ldots . u_{N\alpha}^{*UU}(t))$ is α-open loop saddle point of the Zero-sum model with N participants fuzzy rough continuous-time dynamic game. Where the i^{th} player is interested in maximizing his cost and the remaining players are interested in minimizing i^{th} player cost, if

$$J(u_{1\alpha}^{UU}(t), u_{2\alpha}^{UU}(t), \ldots, u_{(i-1)\alpha}^{UU}(t), u_{i\alpha}^{*UU}(t), u_{(i+1)\alpha}^{UU}(t), \ldots u_{N\alpha}^{UU}(t)) \leq$$
$$J(u_{1\alpha}^{*UU}(t), u_{2\alpha}^{*UU}(t), \ldots, u_{(i-1)\alpha}^{*UU}(t), u_{i\alpha}^{*UU}(t), u_{(i+1)\alpha}^{*UU}(t), \ldots . u_{N\alpha}^{*UU}(t)) \leq$$
$$J(u_{1\alpha}^{*UU}(t), u_{2\alpha}^{*UU}(t), \ldots u_{(i-1)\alpha}^{*UU}(t), u_{i\alpha}^{UU}(t), u_{(i+1)\alpha}^{*UU}(t), \ldots . u_{N\alpha}^{*UU}(t)).$$"

Problem Formulation

Mathematical formulation of Zero-sum model with N participants fuzzy rough continuous-time dynamic game is given by:

$$\min_{\tilde{u}^R_{1,}\tilde{u}^R_{2,}\ldots.\tilde{u}^R_N} max\, \tilde{J}^R\left(\tilde{u}^R_1(t),\tilde{u}^R_2(t),\ldots.\tilde{u}^R_N(t)\right) = \tilde{\varphi}^R\left(\tilde{x}^R\left(t_f\right)\right) + \int_{t_0}^{t_f} \tilde{I}^R(\tilde{x}^R(t),\tilde{u}^R_1(t),\tilde{u}^R_2(t),\ldots.\tilde{u}^R_N(t),t)dt \quad (1)$$

With constraints

$$\dot{\tilde{x}}^R(t) = \tilde{f}^R(\tilde{x}^R(t),\tilde{u}^R_1(t),\tilde{u}^R_2(t),\ldots.\tilde{u}^R_N(t),t);\, \tilde{x}^R(t_0) = \tilde{x}^R_0;\, t \in [t_0,t_f] \quad (2)$$

$$\tilde{J}^R_i\left(\tilde{u}^R_1(t),\tilde{u}^R_2(t),\ldots.\tilde{u}^R_N(t)\right) = -\sum_{\substack{j=1\\ j\neq i}}^{N} \tilde{J}^R_j\left(\tilde{u}^R_1(t),\tilde{u}^R_2(t),\ldots.\tilde{u}^R_N(t)\right) \equiv \tilde{J}^R\left(\tilde{u}^R_1(t),\tilde{u}^R_2(t),\ldots.\tilde{u}^R_N(t)\right) \quad (3)$$

Where $\tilde{u}^R_i \in \tilde{U}^R_i \subset R^{m_i}$,$\tilde{x}^R(t): [t_0,t_f] \rightarrow \mathrm{R}$, $\tilde{J}^R_i\left(\tilde{u}^R_1(t),\tilde{u}^R_2(t),\ldots.\tilde{u}^R_N(t)\right)$and $\tilde{f}^R: R^n \times R^s \times [t_0,t_f] \rightarrow f_0$, $\tilde{I}^R: R^n \times R^s \times [t_0,t_f] \rightarrow f_0$ are the control variables, state trajectories, and cost functions represented within a fuzzy–rough structure. $\tilde{f}^R$and $\tilde{I}^R$are continuous fuzzy rough functions. Applying definitions 2.2 and 2.6, problem (1-3) can be classified into four sub-problems, namely UULP (upper upper α-level problem), LULP (lower upper α-level problem), ULLP (upper lower α-level problem) and LLLP (lower lower α-level problem). We analyze UULP in our discussion with three similar consequences.

UPPER-UPPER α-LEVEL PROBLEM (UULP)

$$\min_{u^{UU}_{1\alpha},u^{UU}_{2\alpha},\ldots.u^{UU}_{N\alpha}} max\, J^{UU}_\alpha\left(u^{UU}_{1\alpha}(t),u^{UU}_{2\alpha}(t),\ldots.u^{UU}_{N\alpha}(t)\right) = \varphi^{UU}_\alpha\left(x^{UU}_\alpha\left(t_f\right)\right) + \int_{t_0}^{t_f} I^{UU}_\alpha(x^{UU}_\alpha(t),u^{UU}_{1\alpha}(t),u^{UU}_{2\alpha}(t),\ldots.u^{UU}_{N\alpha}(t),t)dt \quad (4)$$

With constraints

$$\dot{x}^{UU}_\alpha(t) = f^{UU}_\alpha(x^{UU}_\alpha(t),u^{UU}_{1\alpha}(t),u^{UU}_{2\alpha}(t),\ldots.u^{UU}_{N\alpha}(t),t);\, x^{UU}_\alpha(t_0) = x^{UU}_{0\alpha};\, t \in \left[t_0,t_f\right] \quad (5)$$

$$J^{UU}_{i\alpha}\left(u^{UU}_{1\alpha}(t),u^{UU}_{2\alpha}(t),\ldots.u^{UU}_{N\alpha}(t)\right) = -\sum_{\substack{j=1\\ j\neq i}}^{N} J^{UU}_{j\alpha}\left(u^{UU}_{1\alpha}(t),u^{UU}_{2\alpha}(t),\ldots.u^{UU}_{N\alpha}(t)\right) \equiv J^{UU}_\alpha\left(u^{UU}_{1\alpha}(t),u^{UU}_{2\alpha}(t),\ldots.u^{UU}_{N\alpha}(t)\right) \quad (6)$$

LOWER-UPPER α-LEVEL PROBLEM (LULP)

$$\min_{u^{LU}_{1\alpha},u^{LU}_{2\alpha},\ldots.u^{LU}_{N\alpha}} max\, J^{LU}_\alpha\left(u^{LU}_{1\alpha}(t),u^{LU}_{2\alpha}(t),\ldots.u^{LU}_{N\alpha}(t)\right) = \varphi^{LU}_\alpha\left(x^{LU}_\alpha\left(t_f\right)\right) + \int_{t_0}^{t_f} I^{LU}_\alpha(x^{LU}_\alpha(t),u^{LU}_{1\alpha}(t),u^{LU}_{2\alpha}(t),\ldots.u^{LU}_{N\alpha}(t),t)dt$$

With constraints

$$\dot{x}^{LU}_\alpha(t) = f^{LU}_\alpha(x^{LU}_\alpha(t),u^{LU}_{1\alpha}(t),u^{LU}_{2\alpha}(t),\ldots.u^{LU}_{N\alpha}(t),t);\, x^{LU}_\alpha(t_0) = x^{LU}_{0\alpha};\, t \in \left[t_0,t_f\right]$$

$$J_{i\alpha}^{LU}\left(u_{1\alpha}^{LU}(t), u_{2\alpha}^{LU}(t), \dots . u_{N\alpha}^{LU}(t)\right) = -\sum_{\substack{j=1 \\ j\neq i}}^{N} J_{j\alpha}^{LU}\left(u_{1\alpha}^{LU}(t), u_{2\alpha}^{LU}(t), \dots . u_{N\alpha}^{LU}(t)\right)$$
$$\equiv J_{\alpha}^{LU}\left(u_{1\alpha}^{LU}(t), u_{2\alpha}^{LU}(t), \dots . u_{N\alpha}^{LU}(t)\right)$$

UPPER-LOWER α-LEVEL PROBLEM (ULLP)

$$\min_{u_{1\alpha}^{UL}, u_{2\alpha}^{UL}, \dots . u_{N\alpha}^{UL}} max\, J_{\alpha}^{UL}\left(u_{1\alpha}^{UL}(t), u_{2\alpha}^{UL}(t), \dots . u_{N\alpha}^{UL}(t)\right)$$
$$= \varphi_{\alpha}^{UL}\left(x_{\alpha}^{UL}(t_f)\right) + \int_{t_0}^{t_f} I_{\alpha}^{UL}(x_{\alpha}^{UL}(t), u_{1\alpha}^{UL}(t), u_{2\alpha}^{UL}(t), \dots . u_{N\alpha}^{UL}(t), t)dt \quad (4$$

With constraints

$$\dot{x}_{\alpha}^{UL}(t) = f_{\alpha}^{UL}(x_{\alpha}^{UL}(t), u_{1\alpha}^{UL}(t), u_{2\alpha}^{UL}(t), \dots . u_{N\alpha}^{UL}(t), t); x_{\alpha}^{UL}(t_0) = x_{0\alpha}^{UL}; t \in [t_0, t_f] \quad (5$$
$$J_{i\alpha}^{UL}\left(u_{1\alpha}^{UL}(t), u_{2\alpha}^{UL}(t), \dots . u_{N\alpha}^{UL}(t)\right) = -\sum_{\substack{j=1 \\ j\neq i}}^{N} J_{j\alpha}^{UL}\left(u_{1\alpha}^{UL}(t), u_{2\alpha}^{UL}(t), \dots . u_{N\alpha}^{UL}(t)\right)$$
$$\equiv J_{\alpha}^{UL}\left(u_{1\alpha}^{UL}(t), u_{2\alpha}^{UL}(t), \dots . u_{N\alpha}^{UL}(t)\right)$$

LOWER-LOWER α-LEVEL PROBLEM (LLLP)

$$\min_{u_{1\alpha}^{LL}, u_{2\alpha}^{LL}, \dots . u_{N\alpha}^{LL}} max\, J_{\alpha}^{LL}\left(u_{1\alpha}^{LL}(t), u_{2\alpha}^{LL}(t), \dots . u_{N\alpha}^{LL}(t)\right)$$
$$= \varphi_{\alpha}^{LL}\left(x_{\alpha}^{LL}(t_f)\right) + \int_{t_0}^{t_f} I_{\alpha}^{LL}(x_{\alpha}^{LL}(t), u_{1\alpha}^{LL}(t), u_{2\alpha}^{LL}(t), \dots . u_{N\alpha}^{LL}(t), t)dt$$

With constraints

$$\dot{x}_{\alpha}^{LL}(t) = f_{\alpha}^{LL}(x_{\alpha}^{LL}(t), u_{1\alpha}^{LL}(t), u_{2\alpha}^{LL}(t), \dots . u_{N\alpha}^{LL}(t), t); x_{\alpha}^{LL}(t_0) = x_{0\alpha}^{LL}; t \in [t_0, t_f]$$
$$J_{i\alpha}^{LL}\left(u_{1\alpha}^{LL}(t), u_{2\alpha}^{LL}(t), \dots . u_{N\alpha}^{LL}(t)\right) = -\sum_{\substack{j=1 \\ j\neq i}}^{N} J_{j\alpha}^{LL}\left(u_{1\alpha}^{LL}(t), u_{2\alpha}^{LL}(t), \dots . u_{N\alpha}^{LL}(t)\right)$$
$$\equiv J_{\alpha}^{LL}\left(u_{1\alpha}^{LL}(t), u_{2\alpha}^{LL}(t), \dots . u_{N\alpha}^{LL}(t)\right)$$

Theorem: (Necessary Condition) Let $I_{\alpha}^{UU}(x_{\alpha}^{UU}(t), u_{1\alpha}^{UU}(t), u_{2\alpha}^{UU}(t), \dots u_{N\alpha}^{UU}(t), t)$ and $f_{\alpha}^{UU}(x_{\alpha}^{UU}(t), u_{1\alpha}^{UU}(t), u_{2\alpha}^{UU}(t), \dots u_{N\alpha}^{UU}(t), t)$ are C^1 maps. If $x_{\alpha}^{*UU}(t)$ is $\alpha-$ state trajectory, $\left(u_{1\alpha}^{*UU}(t), u_{2\alpha}^{*UU}(t), \dots . u_{N\alpha}^{*UU}(t)\right)$ is α -open loop saddle point for (UULP) then ∃ a costate vector $P_{\alpha}^{UU}(t): [t_0, t_f] \rightarrow R^n$ and Hamiltonian

$$\begin{aligned} H_{\alpha}^{UU}&(x_{\alpha}^{UU}(t), u_{1\alpha}^{UU}(t), u_{2\alpha}^{UU}(t), \dots u_{N\alpha}^{UU}(t), P_{\alpha}^{UU}(t), t) \\ &= I_{\alpha}^{UU}(x_{\alpha}^{UU}(t), u_{1\alpha}^{UU}(t), u_{2\alpha}^{UU}(t), \dots u_{N\alpha}^{UU}(t), t) \\ &+ P_{\alpha}^{UU}(t)^T f_{\alpha}^{UU}(x_{\alpha}^{UU}(t), u_{1\alpha}^{UU}(t), u_{2\alpha}^{UU}(t), \dots u_{N\alpha}^{UU}(t), t) \end{aligned} \quad (7)$$

With constraints

$$\dot{x}_{\alpha}^{UU}(t) = f_{\alpha}^{UU}(x_{\alpha}^{UU}(t), u_{1\alpha}^{UU}(t), u_{2\alpha}^{UU}(t), \dots u_{N\alpha}^{UU}(t), t); x_{\alpha}^{UU}(t_0) = x_{0\alpha}^{UU} \quad (8)$$

$$\dot{P}_{\alpha}^{UU}(t) = -\frac{\partial H_{\alpha}^{UU}(x_{\alpha}^{UU}(t), u_{1\alpha}^{UU}(t), u_{2\alpha}^{UU}(t), \dots u_{N\alpha}^{UU}(t), P_{\alpha}^{UU}(t), t)}{\partial x_{\alpha}^{UU}} \quad (9)$$

$$P_{\alpha}^{UU}(t_f)^T = \frac{\partial \varphi_{\alpha}^{UU}\left(x_{\alpha}^{UU}(t_f)\right)}{\partial x_{\alpha}^{UU}(t_f)} \quad (10)$$

$$\begin{aligned} &\min_{u_{i\alpha}^{UU}} H_{\alpha}^{UU}(x_{\alpha}^{UU}(t), u_{1\alpha}^{*UU}(t), u_{2\alpha}^{*UU}(t), \dots u_{(i-1)\alpha}^{*UU}(t), u_{i\alpha}^{UU}(t), u_{(i+1)\alpha}^{*UU}(t), \dots u_{N\alpha}^{*UU}(t), P_{\alpha}^{UU}(t \\ &= H_{\alpha}^{UU}(x_{\alpha}^{UU}(t), u_{1\alpha}^{*UU}(t), u_{2\alpha}^{*UU}(t), \dots u_{(i-1)\alpha}^{*UU}(t), u_{i\alpha}^{*UU}(t), u_{(i+1)\alpha}^{*UU}(t), \dots u_{N\alpha}^{*UU}(t), P_{\alpha}^{UU}(t) \\ &= \max_{\substack{u_{j\alpha}^{UU} \\ j\neq i}} H_{\alpha}^{UU}(x_{\alpha}^{UU}(t), u_{1\alpha}^{UU}(t), u_{2\alpha}^{UU}(t), \dots u_{(i-1)\alpha}^{UU}(t), u_{i\alpha}^{*UU}(t), u_{(i+1)\alpha}^{UU}(t), \dots u_{N\alpha}^{UU}(t), P_{\alpha}^{UU} \end{aligned} \quad (11)$$

Further if $\left(u_{1\alpha}^{*UU}(t), u_{2\alpha}^{*UU}(t), \dots . u_{N\alpha}^{*UU}(t)\right)$ is an interior solution, then

$$\frac{\partial H_{\alpha}^{UU}(x_{\alpha}^{UU}(t), u_{1\alpha}^{UU}(t), u_{2\alpha}^{UU}(t), \dots u_{N\alpha}^{UU}(t), P_{\alpha}^{UU}(t), t)}{\partial u_{i\alpha}^{UU}} = 0 \,;\forall\, i = 1, 2, \dots N \quad (12)$$

And

$$\begin{aligned} &\frac{\partial^2 H_{\alpha}^{UU}(x_{\alpha}^{UU}(t), u_{1\alpha}^{UU}(t), u_{2\alpha}^{UU}(t), \dots u_{N\alpha}^{UU}(t), P_{\alpha}^{UU}(t), t)}{\partial^2 u_{j\alpha}^{UU}} \leq 0; \text{ at} \\ &(u_{1\alpha}^{UU}(t), u_{2\alpha}^{UU}(t), \dots u_{(i-1)\alpha}^{UU}(t), u_{i\alpha}^{UU}(t), u_{(i+1)\alpha}^{UU}(t), \dots u_{N\alpha}^{UU}(t)) \\ &\quad = (u_{1\alpha}^{*UU}(t), u_{2\alpha}^{*UU}(t), \dots u_{(i-1)\alpha}^{*UU}(t), u_{i\alpha}^{UU}(t), u_{(i+1)\alpha}^{*UU}(t), \dots u_{N\alpha}^{*UU}(t) \\ &\qquad \forall\, j \neq i \end{aligned} \quad (13)$$

And

$$\begin{aligned} &\frac{\partial^2 H_{\alpha}^{UU}(x_{\alpha}^{UU}(t), u_{1\alpha}^{UU}(t), u_{2\alpha}^{UU}(t), \dots u_{N\alpha}^{UU}(t), P_{\alpha}^{UU}(t), t)}{\partial^2 u_{i\alpha}^{UU}} \geq 0; \text{ at} \\ &(u_{1\alpha}^{UU}(t), u_{2\alpha}^{UU}(t), \dots u_{(i-1)\alpha}^{UU}(t), u_{i\alpha}^{UU}(t), u_{(i+1)\alpha}^{UU}(t), \dots u_{N\alpha}^{UU}(t)) = \\ &(u_{1\alpha}^{UU}(t), u_{2\alpha}^{UU}(t), \dots u_{(i-1)\alpha}^{UU}(t), u_{i\alpha}^{*UU}(t), u_{(i+1)\alpha}^{UU}(t), \dots u_{N\alpha}^{UU}(t)) \end{aligned} \quad (14)$$

Proof: Provided that $I_{\alpha}^{UU}(x_{\alpha}^{UU}(t), u_{1\alpha}^{UU}(t), u_{2\alpha}^{UU}(t), \dots u_{N\alpha}^{UU}(t), t)$ and

$f_{\alpha}^{UU}(x_{\alpha}^{UU}(t), u_{1\alpha}^{UU}(t), u_{2\alpha}^{UU}(t), \dots u_{N\alpha}^{UU}(t), t)areC^1$ maps, then there exists a solution $P_{\alpha}^{UU}(t)$ for the following differential equation

$$\begin{aligned} &\dot{P}_{\alpha}^{UU}(t) \\ &= -\frac{\partial I_{\alpha}^{UU}\left(x_{\alpha}^{UU}(t), u_{1\alpha}^{*UU}(t), u_{2\alpha}^{*UU}(t), \dots u_{(i-1)\alpha}^{*UU}(t), u_{i\alpha}^{UU}(t), u_{(i+1)\alpha}^{*UU}(t), \dots u_{N\alpha}^{*UU}(t), t\right)}{\partial x_{\alpha}^{UU}} \\ &- P_{\alpha}^{UU}(t)^T \frac{\partial f_{\alpha}^{UU}(x_{\alpha}^{UU}(t), u_{1\alpha}^{*UU}(t), u_{2\alpha}^{*UU}(t), \dots u_{(i-1)\alpha}^{8UU}(t), u_{i\alpha}^{UU}(t), u_{(i+1)\alpha}^{*UU}(t), \dots u_{N\alpha}^{*UU}(t), t)}{\partial x_{\alpha}^{UU}} \end{aligned} \quad (15)$$

With adjoint equation

$$P_\alpha^{UU}(t)\delta\dot{x}_\alpha^{UU}(t)$$
$$= \left[\frac{\partial I_\alpha^{UU}\left(x_\alpha^{UU}(t), u_{1\alpha}^{*UU}(t), u_{2\alpha}^{*UU}(t), \dots u_{(i-1)\alpha}^{*UU}(t), u_{i\alpha}^{UU}(t), u_{(i+1)\alpha}^{*UU}(t), \dots u_{N\alpha}^{*UU}(t), t\right)}{\partial x_\alpha^{UU}}\right.$$
$$\left. + P_\alpha^{UU}(t)^T \frac{\partial f_\alpha^{UU}\left(x_\alpha^{UU}(t), u_{1\alpha}^{*UU}(t), u_{2\alpha}^{*UU}(t), \dots u_{(i-1)\alpha}^{*UU}(t), u_{i\alpha}^{UU}(t), u_{(i+1)\alpha}^{*UU}(t), \dots u_{N\alpha}^{*UU}(t), t\right)}{\partial x_\alpha^{UU}}\right]$$
$$+ H_\alpha^{UU}\left(x_\alpha^{UU}(t), u_{1\alpha}^{*UU}(t), u_{2\alpha}^{*UU}(t), \dots u_{(i-1)\alpha}^{*UU}(t), u_{i\alpha}^{*UU}(t), u_{(i+1)\alpha}^{*UU}(t), \dots u_{N\alpha}^{*UU}(t), P_\alpha^{UU}(t), t\right)$$
$$- H_\alpha^{UU}(x_\alpha^{UU}(t), u_{1\alpha}^{*UU}(t), u_{2\alpha}^{*UU}(t), \dots u_{(i-1)\alpha}^{*UU}(t), u_{i\alpha}^{UU}(t), u_{(i+1)\alpha}^{*UU}(t), \dots u_{N\alpha}^{*UU}(t), P_\alpha^{UU}(t), t) \qquad (16)$$

The solution of equation (16) is $\delta x_\alpha^{UU}(t)$ with the initial condition $\delta x_\alpha^{UU}(t_0) = 0$, applying theorem (10.1) in [6]

$$\frac{d}{dt}\left(P_\alpha^{UU}(t)\delta x_\alpha^{UU}(t)\right) = P_\alpha^{UU}(t)\delta\dot{x}_\alpha^{UU}(t) + \dot{P}_\alpha^{UU}(t)\delta x_\alpha^{UU}(t)$$
$$= H_\alpha^{UU}\left(x_\alpha^{UU}(t), u_{1\alpha}^{*UU}(t), u_{2\alpha}^{*UU}(t), \dots u_{(i-1)\alpha}^{*UU}(t), u_{i\alpha}^{*UU}(t), u_{(i+1)\alpha}^{*UU}(t), \dots u_{N\alpha}^{*UU}(t), P_\alpha^{UU}(t), t\right)$$
$$- H_\alpha^{UU}(x_\alpha^{UU}(t), u_{1\alpha}^{*UU}(t), u_{2\alpha}^{*UU}(t), \dots u_{(i-1)\alpha}^{*UU}(t), u_{i\alpha}^{UU}(t), u_{(i+1)\alpha}^{*UU}(t), \dots u_{N\alpha}^{*UU}(t), P_\alpha^{UU}(t), t) \qquad (17)$$

On integrating equation (17) w. r. t. "t" between the limits t_0 to t_f and applying the condition $\delta x_\alpha^{UU}(t_0) = 0$ with $P_\alpha^{UU}(t_f) = \frac{\partial \varphi_\alpha^{UU}(x_\alpha^{UU}(t_f))}{\partial x_\alpha^{UU}(t_f)}$ and $\delta J_\alpha^{UU}\left(u_{1\alpha}^{UU}(t), u_{2\alpha}^{UU}(t), \dots u_{N\alpha}^{UU}(t)\right) = \frac{\partial \varphi_\alpha^{UU}(x_\alpha^{UU}(t_f))}{\partial x_\alpha^{UU}(t_f)}\delta x_\alpha^{UU}(t_f)$

we obtain,

$$\delta J_\alpha^{UU}\left(u_{1\alpha}^{UU}(t), u_{2\alpha}^{UU}(t), \dots u_{N\alpha}^{UU}(t)\right)$$
$$= \int_{t_0}^{t_f} [H_\alpha^{UU}(x_\alpha^{UU}(t), u_{1\alpha}^{*UU}(t), u_{2\alpha}^{*UU}(t), \dots u_{(i-1)\alpha}^{*UU}(t), u_{i\alpha}^{*UU}(t), u_{(i+1)\alpha}^{*UU}(t), \dots u_{N\alpha}^{*UU}(t), P_\alpha^{UU}(t), t)$$
$$- H_\alpha^{UU}(x_\alpha^{UU}(t), u_{1\alpha}^{*UU}(t), u_{2\alpha}^{*UU}(t), \dots u_{(i-1)\alpha}^{*UU}(t), u_{i\alpha}^{UU}(t), u_{(i+1)\alpha}^{*UU}(t), \dots u_{N\alpha}^{*UU}(t), P_\alpha^{UU}(t), t)]\, dt \qquad (18)$$

Applying theorem (11.1) in [6] we obtain,

$$H_\alpha^{UU}(x_\alpha^{UU}(t), u_{1\alpha}^{*UU}(t), u_{2\alpha}^{*UU}(t), \dots u_{(i-1)\alpha}^{*UU}(t), u_{i\alpha}^{*UU}(t), u_{(i+1)\alpha}^{*UU}(t), \dots u_{N\alpha}^{*UU}(t), P_\alpha^{UU}(t), t)$$
$$\leq H_\alpha^{UU}(x_\alpha^{UU}(t), u_{1\alpha}^{*UU}(t), u_{2\alpha}^{*UU}(t), \dots u_{(i-1)\alpha}^{*UU}(t), u_{i\alpha}^{UU}(t), u_{(i+1)\alpha}^{*UU}(t), \dots u_{N\alpha}^{*UU}(t), P_\alpha^{UU}(t), t) \qquad (19)$$

Which gives,

$$\min_{u_{i\alpha}^{UU}} H_\alpha^{UU}(x_\alpha^{UU}(t), u_{1\alpha}^{*UU}(t), u_{2\alpha}^{*UU}(t), \dots u_{(i-1)\alpha}^{*UU}(t), u_{i\alpha}^{*UU}(t), u_{(i+1)\alpha}^{*UU}(t), \dots u_{N\alpha}^{*UU}(t), P_\alpha^{UU}(t), t)$$
$$= H_\alpha^{UU}(x_\alpha^{UU}(t), u_{1\alpha}^{*UU}(t), u_{2\alpha}^{*UU}(t), \dots u_{(i-1)\alpha}^{*UU}(t), u_{i\alpha}^{UU}(t), u_{(i+1)\alpha}^{*UU}(t), \dots u_{N\alpha}^{*UU}(t), P_\alpha^{UU}(t), t) \qquad (20)$$

Similarly,

$$\max_{\substack{u_{j\alpha}^{UU} \\ j \neq i}} H_\alpha^{UU}(x_\alpha^{UU}(t), u_{1\alpha}^{*UU}(t), u_{2\alpha}^{*UU}(t), \dots u_{(i-1)\alpha}^{*UU}(t), u_{i\alpha}^{*UU}(t), u_{(i+1)\alpha}^{*UU}(t), \dots u_{N\alpha}^{*UU}(t), P_\alpha^{UU}(t), t)$$
$$= H_\alpha^{UU}(x_\alpha^{UU}(t), u_{1\alpha}^{*UU}(t), u_{2\alpha}^{*UU}(t), \dots u_{(i-1)\alpha}^{*UU}(t), u_{i\alpha}^{UU}(t), u_{(i+1)\alpha}^{*UU}(t), \dots u_{N\alpha}^{*UU}(t), P_\alpha^{UU}(t), t) \qquad (21)$$

Theorem 3.1.2 (Sufficient Condition) The point $\left(u_{1\alpha}^{*UU}(t), u_{2\alpha}^{*UU}(t), \dots . u_{N\alpha}^{*UU}(t)\right)$ is α-saddle point for the UULP if ∃ a function $P_\alpha^{UU}(t)$ s.t.

[1] $H_\alpha^{UU}\left(x_\alpha^{UU}(t), u_{1\alpha}^{*UU}(t), u_{2\alpha}^{*UU}(t), \dots u_{(i-1)\alpha}^{*UU}(t), u_{i\alpha}^{UU}(t), u_{(i+1)\alpha}^{*UU}(t), \dots u_{N\alpha}^{*UU}(t), P_\alpha^{UU}(t), t\right) - H_\alpha^{UU}\left(x_\alpha^{UU}(t), u_{1\alpha}^{*UU}(t), u_{2\alpha}^{*UU}(t), \dots u_{(i-1)\alpha}^{*UU}(t), u_{i\alpha}^{*UU}(t), u_{(i+1)\alpha}^{*UU}(t), \dots u_{N\alpha}^{*UU}(t), P_\alpha^{UU}(t), t\right) + \dot{P}_\alpha^{UU}(t)(x_\alpha^{UU}(t) - x_\alpha^{*UU}(t)) \geq 0 \forall\, x_\alpha^{UU} \in X_\alpha^{UU}$, $u_\alpha^{UU} \in U_\alpha^{UU}$ and $t \in [t_0, t_f]$.

[2] $P_\alpha^{UU}(t_0)\left(x_\alpha^{UU}(t_0) - x_\alpha^{*UU}(t_0)\right) - P_\alpha^{UU}(t_f)\left(x_\alpha^{UU}(t_f) - x_\alpha^{*UU}(t_f)\right) \geq 0$

Corresponding to inequality

$$J_\alpha^{UU}\left(u_{1\alpha}^{*UU}(t), u_{2\alpha}^{*UU}(t), \dots u_{(i-1)\alpha}^{*UU}(t), u_{i\alpha}^{*UU}(t), u_{(i+1)\alpha}^{*UU}(t), \dots u_{N\alpha}^{*UU}(t)\right) \le J_\alpha^{UU}\left(u_{1\alpha}^{*UU}(t), u_{2\alpha}^{*UU}(t), \dots u_{(i-1)\alpha}^{*UU}(t), u_{i\alpha}^{UU}(t), u_{(i+1)\alpha}^{*UU}(t), \dots u_{N\alpha}^{*UU}(t)\right)$$

[3] $H_\alpha^{UU}\big(x_\alpha^{UU}(t), u_{1\alpha}^{UU}(t), u_{2\alpha}^{UU}(t), \dots u_{(i-1)\alpha}^{UU}(t), u_{i\alpha}^{*UU}(t), u_{(i+1)\alpha}^{UU}(t), \dots u_{N\alpha}^{UU}(t), P_\alpha^{UU}(t), t\big) - H_\alpha^{UU}\big(x_\alpha^{UU}(t), u_{1\alpha}^{*UU}(t), u_{2\alpha}^{*UU}(t), \dots u_{(i-1)\alpha}^{*UU}(t), u_{i\alpha}^{*UU}(t), u_{(i+1)\alpha}^{*UU}(t), \dots u_{N\alpha}^{*UU}(t), P_\alpha^{UU}(t), t\big) + \dot{P}_\alpha^{UU}(t)(x_\alpha^{UU}(t) - x_\alpha^{*UU}(t)) \le 0 \forall\, x_\alpha^{UU} \in X_\alpha^{UU}$, $u_\alpha^{UU} \in U_\alpha^{UU}$ and $t \in [t_0, t_f]$.

[4] $P_\alpha^{UU}(t_0)\big(x_\alpha^{UU}(t_0) - x_\alpha^{*UU}(t_0)\big) - P_\alpha^{UU}(t_f)\left(x_\alpha^{UU}(t_f) - x_\alpha^{*UU}(t_f)\right) \le 0$

corresponding to inequality

$$J_\alpha^{UU}\left(u_{1\alpha}^{UU}(t), u_{2\alpha}^{UU}(t), \dots u_{(i-1)\alpha}^{UU}(t), u_{i\alpha}^{*UU}(t), u_{(i+1)\alpha}^{UU}(t), \dots u_{N\alpha}^{UU}(t)\right) \le J_\alpha^{UU}\left(u_{1\alpha}^{*UU}(t), u_{2\alpha}^{*UU}(t), \dots u_{(i-1)\alpha}^{*UU}(t), u_{i\alpha}^{*UU}(t), u_{(i+1)\alpha}^{*UU}(t), \dots u_{N\alpha}^{*UU}(t)\right)$$

Proof: Let $u_\alpha^{*UU} \in U_\alpha^{UU}$ and for admissible trajectory x_α^{*UU}, Hamiltonian is

$$H_\alpha^{UU}(x_\alpha^{UU}(t), u_{1\alpha}^{UU}(t), u_{2\alpha}^{UU}(t), \dots u_{N\alpha}^{UU}(t), P_\alpha^{UU}(t), t) = I_\alpha^{UU}(x_\alpha^{UU}(t), u_{1\alpha}^{UU}(t), u_{2\alpha}^{UU}(t), \dots u_{N\alpha}^{UU}(t), t) + P_\alpha^{UU}(t)^T f_\alpha^{UU}(x_\alpha^{UU}(t), u_{1\alpha}^{UU}(t), u_{2\alpha}^{UU}(t), \dots u_{N\alpha}^{UU}(t), t) \tag{22}$$

For admissible trajectory x_α^{*UU} and $\dot{x}_\alpha^{UU}(t) = f_\alpha^{UU}(x_\alpha^{UU}(t), u_{1\alpha}^{UU}(t), u_{2\alpha}^{UU}(t), \dots . u_{N\alpha}^{UU}(t), t)$ we have,

$$I_\alpha^{UU}\big(x_\alpha^{*UU}(t), u_{1\alpha}^{*UU}(t), u_{2\alpha}^{*UU}(t), \dots u_{(i-1)\alpha}^{*UU}(t), u_{i\alpha}^{UU}(t), u_{(i+1)\alpha}^{*UU}(t), \dots u_{N\alpha}^{*UU}(t), t\big) - I_\alpha^{UU}\big(x_\alpha^{*UU}(t), u_{1\alpha}^{*UU}(t), u_{2\alpha}^{*UU}(t), \dots u_{(i-1)\alpha}^{*UU}(t), u_{i\alpha}^{*UU}(t), u_{(i+1)\alpha}^{*UU}(t), \dots u_{N\alpha}^{*UU}(t), t\big) + \frac{d}{dt}(P_\alpha^{UU}(t)\big(x_\alpha^{UU}(t) - x_\alpha^{*UU}(t)\big)) \ge 0 \tag{23}$$

On integrating w. r. t. "t" between the limits t_0 to t_f and using the condition (ii), we obtain

$$\int_{t_0}^{t_f}\big[I_\alpha^{UU}\big(x_\alpha^{*UU}(t), u_{1\alpha}^{*UU}(t), u_{2\alpha}^{*UU}(t), \dots u_{(i-1)\alpha}^{*UU}(t), u_{i\alpha}^{UU}(t), u_{(i+1)\alpha}^{*UU}(t), \dots u_{N\alpha}^{*UU}(t), t\big) - I_\alpha^{UU}\big(x_\alpha^{*UU}(t), u_{1\alpha}^{*UU}(t), u_{2\alpha}^{*UU}(t), \dots u_{(i-1)\alpha}^{*UU}(t), u_{i\alpha}^{*UU}(t), u_{(i+1)\alpha}^{*UU}(t), \dots u_{N\alpha}^{*UU}(t), t\big)\big]dt \ge 0 \tag{24}$$

Which gives,

$$J_\alpha^{UU}\left(u_{1\alpha}^{*UU}(t), u_{2\alpha}^{*UU}(t), \dots u_{(i-1)\alpha}^{*UU}(t), u_{i\alpha}^{*UU}(t), u_{(i+1)\alpha}^{*UU}(t), \dots u_{N\alpha}^{*UU}(t)\right) \le J_\alpha^{UU}\left(u_{1\alpha}^{*UU}(t), u_{2\alpha}^{*UU}(t), \dots u_{(i-1)\alpha}^{*UU}(t), u_{i\alpha}^{UU}(t), u_{(i+1)\alpha}^{*UU}(t), \dots u_{N\alpha}^{*UU}(t)\right) \tag{25}$$

Similarly, conditions (iii) and (iv) gives

$$J_\alpha^{UU}\left(u_{1\alpha}^{UU}(t), u_{2\alpha}^{UU}(t), \dots u_{(i-1)\alpha}^{UU}(t), u_{i\alpha}^{*UU}(t), u_{(i+1)\alpha}^{UU}(t), \dots u_{N\alpha}^{UU}(t)\right) \le J_\alpha^{UU}\left(u_{1\alpha}^{*UU}(t), u_{2\alpha}^{*UU}(t), \dots u_{(i-1)\alpha}^{*UU}(t), u_{i\alpha}^{*UU}(t), u_{(i+1)\alpha}^{*UU}(t), \dots u_{N\alpha}^{*UU}(t)\right) \tag{26}$$

Numerical Example

Suppose the state equation and the fuzzy cost function are given by

$$\tilde{\dot{x}}_1^R(t) = \tilde{1}^R\tilde{u}^R(t) + \tilde{1}^R\tilde{v}^R(t) + \tilde{1}^R\tilde{w}^R(t)$$

$$\dot{\tilde{x}}_2^R(t) = \tilde{1}^R\tilde{u}^R(t) + \tilde{1}^R\tilde{v}^R(t) - \tilde{1}^R\tilde{w}^R(t)$$
$$\dot{\tilde{x}}_3^R(t) = \tilde{1}^R\tilde{u}^R(t) - \tilde{1}^R\tilde{v}^R(t) + \tilde{1}^R\tilde{w}^R(t)$$
$$\min_{\tilde{u}^R,\tilde{v}^R,\tilde{w}^R} max \tilde{J}^R(\tilde{u}^R(t),\tilde{v}^R(t),\tilde{w}^R(t))$$
$$= \tilde{1}^R\tilde{x}_1^R(t_f) + \tilde{2}^R\tilde{x}_2^R(t_f) + \tilde{3}^R\tilde{x}_3^R(t_f)$$
$$+ \int_{t_0}^{t_f}\left[((\tilde{2}^R + \tilde{3}^R) - (\tilde{2}^R + \tilde{3}^R)\tilde{u}^R(t))^2 + (\tilde{1}^R - \tilde{1}^R\tilde{v}^R(t))^2\right.$$
$$\left.+ (\tilde{1}^R - \tilde{1}^R\tilde{w}^R(t))^2\right]dt$$

Where the fuzzy numbers $\tilde{1}^R, \tilde{2}^R, \tilde{3}^R$ are given below
$\tilde{1}^R = [(2.5,3,3.5):(2,4,5)]$; $\tilde{2}^R = [(1.5,3,3.5):(1,3,5)]$; $\tilde{3}^R = [(1.5,2.5,3):(1,3,4)]$

Using definition 2.6 and α-cut of triangular fuzzy number, the corresponding UULP is given by, at $\alpha = 0.4$

$$(\dot{x}_1(t))_{0.4}^{UU} = 4.6(u(t))_{0.4}^{UU} + 4.6(v(t))_{0.4}^{UU} + 4.6(w(t))_{0.4}^{UU}$$
$$(\dot{x}_2(t))_{0.4}^{UU} = 4.6(u(t))_{0.4}^{UU} + 4.6(v(t))_{0.4}^{UU} - 2.8(w(t))_{0.4}^{UU}$$
$$(\dot{x}_3(t))_{0.4}^{UU} = 4.6(u(t))_{0.4}^{UU} - 2.8(v(t))_{0.4}^{UU} + 4.6(w(t))_{0.4}^{UU}$$

$$\min_{u_{0.4}^{UU},v_{0.4}^{UU},w_{0.4}^{UU}} max J_{0.4}^{UU}((u(t))_{0.4}^{UU},(v(t))_{0.4}^{UU},(w(t))_{0.4}^{UU})$$
$$= 4.6(x_1(t_f))_{0.4}^{UU} + 4.2(x_2(t_f))_{0.4}^{UU} + 3.6(x_3(t_f))_{0.4}^{UU}$$
$$+ \int_{t_0}^{t_f}[(7.8 - 3.6(u(t))_{0.4}^{UU})^2 + (4.6 - 2.8(v(t))_{0.4}^{UU})^2$$
$$+ (4.6 - 2.8(w(t))_{0.4}^{UU})^2]dt$$

Equations (7-12) yields,

$$(P_1(t_f))_{0.4}^{UU} = 4.6; (P_2(t_f))_{0.4}^{UU} = 4.2; (P_3(t_f))_{0.4}^{UU} = 3.6;$$
$$(u(t))_{0.4}^{UU} = \frac{-0.44}{(3.6)^2}; (v(t))_{0.4}^{UU} = \frac{-2.32}{(2.8)^2}; (w(t))_{0.4}^{UU} = \frac{-0.1}{(2.8)^2}$$

We have, state trajectories,

$$(x_1(t))_{0.4}^{UU} = -1.5761(t_f - t_0); (x_2(t))_{0.4}^{UU} = -1.4817(t_f - t_0);$$
$$(x_3(t))_{0.4}^{UU} = 0.6137(t_f - t_0)$$

In a similar manner we can obtain the controls and the state trajectories associated with each player for the LULP

$$(u(t))_{0.4}^{LU} = \frac{19.12}{(7.8)^2}; (v(t))_{0.4}^{LU} = \frac{10.58}{(4.6)^2}; (w(t))_{0.4}^{LU} = \frac{10.58}{(4.6)^2}; (x_1(t))_{0.4}^{LU} = 3.6799(t_f - t_0); (x_2(t))_{0.4}^{LU} = -0.0201(t_f - t_0); (x_3(t))_{0.4}^{LU} = -0.0201(t_f - t_0)$$

For ULLP

$$(u(t))_{0.4}^{UL} = \frac{8.89}{(4)^2}; (v(t))_{0.4}^{UL} = \frac{1.8}{(2.7)^2}; (w(t))_{0.4}^{UL} = \frac{3.3}{(2.7)^2}; (x_1(t))_{0.4}^{UL} = 4.1422(t_f - t_0);$$
$$(x_2(t))_{0.4}^{UL} = 1.4262(t_f - t_0); (x_3(t))_{0.4}^{UL} = 2.6607(t_f - t_0)$$

For LLLP

$$(u(t))_{0.4}^{LL}=\frac{15.355}{(6.1)^2};\ (v(t))_{0.4}^{LL}=\frac{5.565}{(3.3)^2};\ (w(t))_{0.4}^{LL}=\frac{6.165}{(3.3)^2};(x_1(t))_{0.4}^{LL}=4.0224(t_f-t_0);$$
$$(x_2(t))_{0.4}^{LL}=0.6257(t_f-t_0);\ (x_3(t))_{0.4}^{LL}=0.9563(t_f-t_0)$$

Hence the α-saddle point (for $\alpha=0.4$) of this example is

$$(\tilde{u}^R)_{0.4}=[\left[\frac{15.355}{(6.1)^2},\frac{8.89}{(4)^2}\right]:\left[\frac{19.12}{(7.8)^2},\frac{-0.44}{(3.6)^2}\right]];\ (\tilde{v}^R)_{0.4}=[\left[\frac{5.565}{(3.3)^2},\frac{1.8}{(2.7)^2}\right]:\left[\frac{10.58}{(4.6)^2},\frac{-2.32}{(2.8)^2}\right]]$$
$$(\widetilde{w}^R)_{0.4}=[\left[\frac{6.165}{(3.3)^2},\frac{3.3}{(2.7)^2}\right]:\left[\frac{10.58}{(4.6)^2},\frac{-0.1}{(2.8)^2}\right]]$$

Conclusion

The study establishes the necessary and sufficient criteria for the existence of an α\alphaα-open-loop saddle point. In this work, we analyze an NNN-player zero-sum continuous dynamic game formulated under the fuzzy–rough framework. The derived conditions are illustrated through a numerical example involving three players, evaluated at α = 0.4 under various scenarios. The findings are applicable to situations where it is desirable to optimize the advantage for a specific individual or organization

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